

Positronium as an example of algebraic composite calculations

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Abstract

The functional quantum field theory, developed by Stumpf, provides the possibility to derive the quantum dynamics of a positronium gas from Coulomb interacting electrons and positrons. By this example, the method will be brought in a Clifford algebraic light, through identifying the functional space with an infinite dimensional Euclidean Clifford algebra.

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1 Weak Mapping Formalism of QED

1.1 Motivation

If one likes to describe physics by atomistic theories, for instance the standard model in elementary particle physics, then one has to provide a theory of composites. This is necessary in order to theoretically build up mesons, molecules etc., i.e. the atomistic hierarchy. As the standard model is a parameterization of nature with a large number of parameters and “elementary” fields, it is obvious that such a theory needs a further explanation from first principles, which again leads to composites.

Current quantum field theoretic methods are based on perturbation theory. The most outstanding success of such perturbative methods are the $g - 2$ -factor and the Lamb shift. However calculations where composites are involved can *not* be done in a perturbative manner, because of the strong coupling of the constituents. So other methods have been developed. For example the operator product method [1]. This method considers a pair of operators as a composite. This contradicts the field theoretic point of view, that such a construct should be composed by an infinity of sub-particles generated from the vacuum polarization cloud. In addition, quantum numbers of states can not be uniquely assigned to operator products beyond perturbation theory. Another method to describe composites are Bethe–Salpeter like calculations [2]. These calculations are based on perturbation theory and can not be used to calculate for example excited states for the hydrogen atom [3]. Even if one would succeed to calculate the correct energy levels, one has not achieved a dynamical theory of the bound states.

In this note, we want to outline the quantum field theoretic method of “weak mapping”, which was developed in order to non perturbatively treat the composite particle effective dynamics. We use as an example the Quantum–Electro–Dynamics (QED). This method is formulated in terms of quantum field theoretic quantities and makes use of the technique of generating functionals [4]. However, besides the necessity of giving a short outline of the method, we are mostly interested in the connection of quantum field theories to Clifford algebras. The link between these quite different schemes are the generating functionals. The quantum field theoretical approach involves operators, commutators, propagators, states etc. and tries to give a consistent scheme based on them. The Clifford algebra consists from the very beginning of geometrical entities and geometrical relations, which provide a built in interpretation in geometrical terms. The task of a transition between these pictures is to translate the quantum field theoretic considerations into purely geometric ones and vice versa. It is the hope, that it will become possible to give an algebraic definition of the composite, rather than the heuristic ones used up to now. Therefore it is very useful to have the weak mapping theory available in the algebraic picture.

1.2 QED in Transverse Coordinates

Since QED is a gauge theory, we have to deal with a constrained theory. Especially on the quantum level this is a difficult task [5]. Hence in a first step we will eliminate the gauge freedom and perform the quantization afterwards. This is only possible in QED, because of the simple Abelian gauge coupling. In non abelian Yang–Mills Theories as QCD one has a nonlinear term in the Gauss’ law, and a classical elimination leads to difficulties.

Thus we start with the equations of classical spinor ED

$$i) \quad \partial_\nu F^{\mu\nu} + \frac{ie_0}{2} \Psi C \gamma^\mu \sigma^2 \Psi = 0$$

$$\begin{aligned}
ii) \quad & (i\gamma^\mu \partial_\mu - m_0)\Psi + e_0 A_\mu \gamma^\mu \sigma^3 \Psi = 0 \\
iii) \quad & F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu,
\end{aligned} \tag{1}$$

where we have used the definition

$$\Psi = \Psi_{\alpha\Lambda} = \begin{cases} \Psi_\alpha & \Lambda = 1 \\ \Psi_\alpha^c = C_{\alpha\beta} \bar{\Psi}_\beta & \Lambda = 2 \end{cases} \tag{2}$$

and omit the matrix indices. By definition the sigma matrices act on the super index Λ distinguishing the field and its charge conjugated field, which has to be treated as an independent quantity. The gamma matrices and the charge conjugation matrix C act on the spinor index α of the Ψ field.

Because we are interested in the dynamics, the Hamilton formalism is convenient [6]. This breaks the explicit Lorentz invariance of the theory. But as we are not approximating anything, the theory is still invariant [7]. The bound state calculation requires an explicit representation of the binding force. Hence we use the Coulomb gauge to eliminate the superfluous coordinates. Thus we set

$$\partial_k A^k = 0 \quad k \in \{1, 2, 3\}. \tag{3}$$

This enables us to split the ‘bosonic’ fields A and E into longitudinal and transverse parts.

$$\begin{aligned}
A_k &\equiv A_k^{tr}, & \text{because of } \partial_k A^k &= 0 \\
E_k &= E_k^{tr} + E_k^l, & E_k &:= -F^{0k} = F_{0k}.
\end{aligned} \tag{4}$$

Furthermore we decompose the field equations into dynamic ones and the remaining constraint, the Gauss’ law. Looking at (1-i) yields for

$$\begin{aligned}
\nu = 0 : \quad & i\partial_0 E_k = -\frac{1}{2}e_0 \Psi C \gamma^k \sigma^2 \Psi + i(\partial_j \partial^k A^j - \partial_j \partial^j A^k) \\
\nu = k : \quad & \partial^k E_k = -i\frac{e_0}{2} \Psi C \gamma_0 \sigma^2 \Psi
\end{aligned} \tag{5}$$

and from the definition of E^k

$$i\partial_0 A^{tr,k} = iE^k - i\partial_k A^0. \tag{6}$$

With help of the current conservation the Gauss’ law ($5-\nu = k$) can now be used to yield an expression for the A_0 field

$$A_0 = -i\frac{e_0}{2} \Delta^{-1} \Psi C \gamma_0 \sigma^2 \Psi. \tag{7}$$

The Δ^{-1} symbol is used for the integral kernel of the inverse of the Laplacian. It is now possible to formulate the classical spinor ED in generalized, i.e. in independent, coordinates. For a further compactifying of the notation we introduce a super field B_η also in the ‘bosonic’ fields,

$$B_k^\eta := \begin{cases} A_k^{tr} & \eta = 1 \\ E_k^{tr} & \eta = 2 \end{cases} . \quad (8)$$

The equations of motion then read compactified

$$\begin{aligned} i\partial_0 \Psi_{I_1} &= D_{I_1 I_2} \Psi_{I_2} + W_{I_1 I_2}^k B_k \Psi_{I_2} + U_{I_1}^{I_2 I_3 I_4} \Psi_{I_2} \Psi_{I_3} \Psi_{I_4} \\ i\partial B_{K_1} &= L_{K_1 K_2} B_{K_2} + J_{K_1}^{I_1 I_2} \Psi_{I_1} \Psi_{I_2} \end{aligned} \quad (9)$$

with the definitions

$$\begin{aligned} I &:= \{\alpha, \Lambda, \mathbf{r}\} \\ K &:= \{k, \eta, \mathbf{z}\} \\ P^{tr} &:= 1 - \Delta^{-1} \nabla \otimes \nabla \\ D_{I_1 I_2} &:= -(i\gamma_0 \gamma^k \partial_k - \gamma_0 m)_{\alpha_1 \alpha_2} \delta_{\Lambda_1 \Lambda_2} \delta(\mathbf{r}_1 - \mathbf{r}_2) \\ W_{I_1 I_2}^K &:= e_0 (\gamma_0 \gamma^k)_{\alpha_1 \alpha_2} \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{z}) \delta_{1\eta} \sigma_{\Lambda_1 \Lambda_2}^3 \\ U_{I_1}^{I_2 I_3 I_4} &:= -\frac{i}{8\pi} e_0^2 \left[(C\gamma_0)_{\alpha_2 \alpha_3} \delta_{\alpha_1 \alpha_4} \sigma_{\Lambda_2 \Lambda_3}^2 \sigma_{\Lambda_1 \Lambda_4}^2 \frac{\delta(\mathbf{r}_2 - \mathbf{r}_3) \delta(\mathbf{r}_1 - \mathbf{r}_4)}{|\mathbf{r}_1 - \mathbf{r}_2|} \right]_{as\{I_2 I_3 I_4\}} \\ L_{K_1 K_2} &:= i\delta(\mathbf{z}_2 - \mathbf{z}_1) \delta_{k_1 k_2} \delta_{\eta_1 1} \delta_{\eta_2 2} + i\Delta(\mathbf{z}_1) \delta(\mathbf{z}_1 - \mathbf{z}_2) \delta_{k_1 k_2} \delta_{\eta_1 2} \delta_{\eta_2 1} \\ J_K^{I_1 I_2} &:= -\frac{1}{2} e_0 P^{tr}(\mathbf{z} - \mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) (C\gamma^k)_{\alpha_1 \alpha_2} \delta_{2\eta} \sigma_{\Lambda_1 \Lambda_2}^2 . \end{aligned} \quad (10)$$

Now we want to perform the quantization. We therefore impose commutation relations and the Ψ and B fields become operators acting on a suitable state space. These states should belong to an Hilbert space, if the theory would be renormalized. The topic of renormalization is beyond the scope of this note, and was discussed in [8] for bound states in a perturbative manner. As a postulate we introduce the equal time commutation relations to be

$$\begin{aligned} i) \quad & \{\Psi_{I_1}, \Psi_{I_2}\}_+^t := A_{I_1 I_2} = C\gamma_0 \sigma^1 \delta(\mathbf{r}_1 - \mathbf{r}_2) \\ ii) \quad & [B_K, \Psi_I]_-^t := 0 \\ iii) \quad & [B_{K_1}, B_{K_2}]_-^t := C_{K_1 K_2} . \end{aligned} \quad (11)$$

Equation (11-i) is nothing but the canonical commutation relation for Ψ_α and $\bar{\Psi}_\alpha$, the Dirac adjoint field. In our notation with charge conjugate spinor operators, the somehow unusual $A_{I_1 I_2}$ occurs. Because of the super field notation we have all four commutators belonging to the Ψ_α and $\bar{\Psi}_\alpha$ integrated in this relation.

Equation (11-ii) states, that the boson fields are looked at as elementary fields. That is, the B is *not* a function of the Ψ_I fields. The B field is especially *not* composed of an even number of Ψ_I fields, because these Ψ_I 's should then be out of a disjunct index set $\{I'\}$. This would be a contradiction, because the relation should hold for all indices I . Equation (11-ii) is usual postulated in QED.

Relation (11-iii) specifies only the grading of the boson commutator. That is, the (boson) grade or boson number is lowered by two through commutation. The result is a generalized c-number function, thus a complex valued distribution. The special form of the function $C_{K_1 K_2}$ can be calculated as a consequence of requiring a consistent quantum field theory [8], see below (17).

The set of equations (9) and (11) are the defining relations of Coulomb gauged quantized spinor QED. All previous steps are classical and only for convenience to state a somehow selfconsistent QFT. If one would quantize first and then try to eliminate the Gauss' law, one would yield an other QFT! The functional Hamiltonian (see below) would then have infinite many terms [9].

1.3 Generating Functionals

From the Wightman reconstruction theorem [10] it follows that if all vacuum expectation values are known, then the field theory can be reconstructed. Thus one has to give equations for all such matrix elements. This leads to the Dyson–Schwinger–Freese equations, for covariant timeordered matrix elements, which so far can be only derived by perturbation theory.

If, however, one wants to have a complete quantum field theory with state space construction and nonperturbative dynamics, this implies the necessity of using the Hamilton formalism. Therefore we use one–time functionals exclusively. In our one–time formalism we can introduce separate orderings for both kinds of fields.

$$\tau_{nm}^t(a) := \langle 0 | \mathcal{A}(\Psi_{I_1}^t \dots \Psi_{I_n}^t) \mathcal{S}(B_{K_1}^t \dots B_{K_m}^t) | a \rangle. \quad (12)$$

Here $\langle 0 |$ is the physical vacuum state. Thus both Ψ_α and Ψ_α^c do not annihilate in general this state! The Fock state is related to the free case without interaction, and only perturbative methods may use it. We want to calculate bound states, so a strong interaction contradicts the Fock state.

Because we are interested especially in energy values, we use $|a\rangle$ as an energy eigenstate. $|0\rangle$ is a special case in which we are able to use Wightman’s theorem. In the energy representation we are able to separate the time dependence.

The (anti) symmetrization in the field operators corresponds to the Hamilton formalism, because this would be the correct onetime limiting case if we would deal with a single class of fields only. To collect the whole set of transition matrix elements into one expression, we define functional sources j, b and the corresponding contractions ∂, ∂^b with the relations

$$\begin{aligned} \{\partial_{I_1}, j_{I_2}\}_+ &= \delta_{I_1 I_2} \\ [\partial_{K_1}^b, b_{K_2}]_- &= \delta_{K_1 K_2} \\ \partial_I |0\rangle_f &= \partial_K^b |0\rangle_f = 0 \\ {}_f \langle 0 | j_I &= {}_f \langle 0 | b_K = 0. \end{aligned} \quad (13)$$

(Not displayed (anti) commutators are zero.) The functional Fock space construction $|0\rangle_f$ is independent of the physical vacuum. Now we are able to build the formal construct of a functional state [11]

$$\begin{aligned} |\mathbf{A}(a, j, b)\rangle &:= \sum_{i=0}^{\infty} \frac{i^n}{n!m!} \tau_{nm}^t(a) j_{I_1}^t \dots j_{I_n}^t b_{K_1}^t \dots b_{K_m}^t |0\rangle_f \\ \tau_{nm}^t(a) &= \langle 0 | \mathcal{A}(\Psi_{I_1}^t \dots \Psi_{I_n}^t) \mathcal{S}(B_{K_1}^t \dots B_{K_m}^t) | a \rangle \end{aligned} \quad (14)$$

where we omit in further formulas the index t for the sake of brevity. The transition matrix elements are recovered by ‘projecting’ from the left with the contractions ∂_I and ∂_K^b and setting the sources zero, which is equivalent to the action of ${}_f \langle 0 |$ from the left

$$\tau_{nm}^t(a) = \frac{1}{i^n} {}_f \langle 0 | \partial_{K_1}^b \dots \partial_{K_m}^b \partial_{I_1} \dots \partial_{I_n} | \mathbf{A}(a, j, b) \rangle. \quad (15)$$

Hence one functional state comprises the whole hierarchy, which can be explicitly recovered by such a projection technique.

1.4 Dynamics in the Functional Space

The next step is to implement the dynamics into the functional formalism. First of all, we have the Hamiltonian corresponding to the equations (9) as

$$\begin{aligned}
H(\Psi, B) = & \frac{1}{2} A_{I_1 I_3} D_{I_3 I_2} \Psi_{I_1} \Psi_{I_2} + \frac{1}{2} A_{I_1 I_3} W_{I_3 I_2}^K B_K \Psi_{I_1} \Psi_{I_2} \\
& + \frac{1}{4} A_{I_1 I_5} U_{I_5}^{I_2 I_3 I_4} \Psi_{I_1} \Psi_{I_2} \Psi_{I_3} \Psi_{I_4} \\
& + \frac{1}{2} C_{K_1 K_3} L_{K_3 K_2} B_{K_1} B_{K_2}.
\end{aligned} \tag{16}$$

There is no term $J_K^{I_1 I_2}$ in the Hamiltonian. This term was eliminated with help of the commutator (11–ii). Taking the time derivative and using the dynamics, we are left with

$$C_{K_1 K} W_{I_1 I_2}^K \Psi_{I_2} = 2 A_{I_1 I} J_K^{I I_2} \Psi_{I_2}. \tag{17}$$

This requirement, which is in fact an additional assumption, that this equation holds for every Ψ_I allows to drop the Ψ . Of course Ψ is *not* an invertible operator! This sort of consistency relation may be motivated by the physical assumption, that there is no asymmetry in the boson–fermion and fermion–boson interaction (actio equals reactio, Newton). Furthermore, one has two possibilities of multiplications of fields operators to antisymmetric groups of them, that is from right or left. The requirement, that the result should not depend on that choice also leads to the above relation [8].

As a consequence of the introduction of generating functional states we have to transform (16) into this space

$$H[\Psi, B] \implies H[j, b, \partial, \partial^b]. \tag{18}$$

If we choose $|a\rangle$ as energy eigenstate, this results in the functional Schrödinger equation

$$i\partial_0 |\mathbf{A}(a, j, b)\rangle = E_{0a} |\mathbf{A}(a, j, b)\rangle = H[j, b, \partial, \partial^b] |\mathbf{A}(a, j, b)\rangle. \tag{19}$$

In general the matrix elements are transition matrix elements, and thus not directly related to a statistical interpretation. The connection to such an interpretation, i.e. the map of the (positive definite) Hilbert space into functional space, was investigated in [12].

We are exclusively interested in one time functionals, and may derive the time derivative by application of the Baker–Campbell–Hausdorff formula [13, 14]

$$\begin{aligned}
i\partial_0 e^{A(t)} &= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} [H, A]_k \\
[H, A]_k &:= [H, [H, A]_{k-1}]; \quad k \geq 1 \\
[H, A]_0 &:= i\partial_0 A = [H, A].
\end{aligned} \tag{20}$$

One has to calculate the iterated commutators. Because of the grade lowering character of the commutators and the commutativity of bosons and fermions, this process is finite. The highest term

is of order four. The result is an equation involving functional sources and field operators, i.e. a mixture of old and new variables

$$H[\Psi, B] \implies H[j, b, \Psi, B]. \quad (21)$$

To eliminate the field operators from this expression, one has to use the Baker–Campbell–Hausdorff formula once more, with respect to the functional derivatives (contractions). The final result is

$$\begin{aligned} E_{0a}|\mathbf{A}(a, j, b) > &= \left\{ D_{I_1 I_2} j_{I_1} \partial_{I_2} + W_{I_1 I_2}^K j_{I_1} \partial_{I_2} \partial_K^b + L_{K_1 K_2} b_{K_1} \partial_{K_2}^b \right. \\ &\quad + U_{I_1}^{I_2 I_3 I_4} j_{I_1} (\partial_{I_2} \partial_{I_3} \partial_{I_4} - \frac{1}{4} A_{I_3 I'_3} A_{I_2 I'_2} j_{I'_2} j_{I'_3} j_{I_4}) \\ &\quad \left. + J_K^{I_1 I_2} b_K (\partial_{I_1} \partial_{I_2} + \frac{1}{4} A_{I_1 I'_1} A_{I_2 I'_2} j_{I'_1} j_{I'_2}) \right\} |\mathbf{A}(a, j, b) >, \end{aligned} \quad (22)$$

where the $A_{I_n I_m}$ terms stem from the field quantization of the fermions. Boson quantization terms would again occur, if the $J_K^{I_1 I_2}$ would be eliminated in the above mentioned way.

Summarizing we have obtained the Hamilton formalism transformed into functional space. The use of the Hamilton formalism is imperative in order to preserve scalar products and normalization, which is needed for the derivation of a composite particle dynamics.

1.5 Definition of the Composite

Now we want to define composites to build up positronium states. Because of the elimination of the superfluous degrees of freedom, the remaining bosonic coordinates B_K are transverse. They describe “real” photons, in the language of QFT. The “virtual” photons are contained in the scalar part, which was eliminated in favor of the (inter fermionic) Coulomb interaction. Thus we may restrict ourselves to the fermionic part of the functional equation. This is easily done by setting the bosonic sources zero. The remaining equation is a nonlinear spinor field equation with a nonlocal self coupling.

Before we can proceed, we have to eliminate uncorrelated parts from the matrix elements. Because we are interested in two particle bound states, it is sufficient to remove the uncorrelated parts only up to this stage. This can be achieved by the functional normal ordering [4]. We thus define a new functional state

$$|\mathbf{A}(a, j, b) > =: e^{-\frac{1}{2}jFj} |\mathbf{F}(a, j, b) >, \quad (23)$$

where F is the propagator of the theory and the exponential is an abbreviation for the formal series.

Inserting this equation into (22), commuting the exponential factor to the left and multiplying with the inverse from the left, we get the normal ordered equation. This normal ordering is nonperturbative and *not* related to Fock space. But in the case of Fock space one obtains the same result as in the usual normal ordering [15], which justifies the name. We have

$$\begin{aligned} d_I &:= \partial_I - F_{II'} j_{I'} \\ H[j, b, \partial, \partial^b] e^{-\frac{1}{2}jFj} &= e^{-\frac{1}{2}jFj} H[j, b, d, \partial^b] \\ &= e^{-\frac{1}{2}jFj} \hat{H}[j, b, \partial, \partial^b] \end{aligned} \quad (24)$$

and then (see below (67))

$$E_{0a}|\mathbf{F}(a, j, b) > = \hat{H}[j, b, \partial, \partial^b]|\mathbf{F}(a, j, b) > . \quad (25)$$

So far we have normal transformed the full functional equation. Now we close the bosonic channels in order to define bound states. This is necessary, because of the meta stable nature of the positronium states, which would not allow to perform such a calculation.

Def.: *The composite is a solution of the diagonal part of the functional equation for $b_K, \partial_K^b \equiv 0$. The diagonal part consists of an equal number of fermionic sources and contractions without the renormalization terms.*

By this definition the diagonal equation results as

$$\begin{aligned} E_{0a}|\mathbf{F}(a, j, b) > &= \hat{H}^d|\mathbf{F}(a, j, b) > \\ \hat{H}^d &\approx j\partial, jj\partial\partial, \dots \end{aligned} \quad (26)$$

In configuration space this equation decomposes into a set of decoupled equations for various lepton numbers. For the positronium states, we restrict ourselves to the two particle sector. The formal solution is denoted by

$$C_k^{I_1 I_2} := \langle 0 | \mathcal{A}(\Psi_{I_1} \Psi_{I_2}) | a \rangle, \quad (27)$$

which solves (26). k describes the quantum numbers of the bound state with respect to the state $|a\rangle$. The equations for the diagonal part are very similar to Bethe–Salpeter like equations.

1.6 Map onto the Effective Theory

The “weak mapping” is now a rewriting of the normal ordered functional equation. This can be achieved by exact mapping theorems [16]. The resulting equations describe a positronium gas coupled to the electrons, positrons and photons. All exchange forces and quantum effects are accounted for.

Here we want to use a short cut method, which is much more simpler. We neglect thereby several exchange forces, but in the low energy regime and for small positronium concentrations these terms become small corrections. A further simplification will be the restriction to the even part of the functional equation. This can be done, because the even and odd parts decouple. We are thus left with the positronium gas and the photon field, as well as with their coupling and quantum effects.

In the next step we define a set of dual functions to the positronium solutions $C_k^{I_1 I_2}$ by

$$\begin{aligned} R_{I_1 I_2}^k C_{k'}^{I_1 I_2} &= \delta_{k'}^k \\ R_{I_1 I_2}^k C_k^{I_3 I_4} &= \delta_{\{I_1 I_2\}_{as}}^{\{I_3 I_4\}}. \end{aligned} \quad (28)$$

This duality relation is only possible, if the solutions $C_k^{I_1 I_2}$ span the whole space of two-particles solutions. We have thus to include scattering states also, which occur if the rest frame energy increases over a certain threshold. If one would like to omit these scattering states by the introduction of an energy cut-off, one would have to project onto a subspace. However, this would also restrict the possible physical situations after the mapping.

By means of (28) we introduce new (independent) sources for the positronium states

$$c_k := \frac{1}{2} C_k^{I_1 I_2} j_{I_1} j_{I_2}. \quad (29)$$

The main problem to solve in the above mentioned weak mapping theorems is the correct definition of the sources c_k as independent quantities and the correct definition of the corresponding contraction. This can be achieved exactly. For a rewriting we may also use, as a short cut, the functional chain rule, given by

$$\partial_{I_1} = \frac{\partial c_k}{\partial j_{I_1}} \partial_k^c = C_k^{I_1 I_2} j_{I_2} \partial_k^c. \quad (30)$$

With help of the duals (28), the source definition (29) and the chain rule (30), we can rewrite the normal ordered functional Hamiltonian

$$\hat{H}[j, b, \partial, \partial^b] \cong H[c, b, \partial^c, \partial^b]. \quad (31)$$

We have to rewrite also the functional state $|\mathbf{F}(a, j, b) \rangle$ even to $|\mathbf{F}(a, c, b) \rangle$. The resulting equation is

$$\begin{aligned} E_{0a} |\mathbf{F}(a, c, b) \rangle &= \left\{ R_{I_1 I_2}^{k_1} D_{I_1 I_3} C_{k_2}^{I_3 I_2} c_{k_1} \partial_{k_2}^c - R_{I_1 I_2}^k Z_{I_1 I_3} F_{I_3 I_2} c_k + L_{K_1 K_2} b_{K_1} \partial_{K_2}^b \right. \\ &+ W_{I_1 I_2}^K [R_{I_1 I_3}^{k_1} C_{k_2}^{I_2 I_3} c_{k_1} \partial_{k_2}^c \partial_K^b - R_{I_1 I_3}^k F_{I_2 I_3} c_k \partial_K^b] \\ &- U_J^{I_1 I_2 I_3} \left[4 R_{\{J I_4}^{k_1} R_{I_5 I_6\} a s}^{k_2} C_{k_1'}^{I_1 I_4} C_{k_2'}^{I_2 I_5} C_{k_3'}^{I_3 I_6} c_{k_1} c_{k_2} \partial_{k_1'}^c \partial_{k_2'}^c \partial_{k_3'}^c \right. \\ &- 3 R_{J I_4}^k C_{k_1'}^{I_1 I_3} C_{k_2'}^{I_2 I_4} c_k \partial_{k_1'}^c \partial_{k_2'}^c + 12 R_{\{J I_4}^{k_1} R_{I_5 I_6\} a s}^{k_2} F_{I_3 I_4} C_{k_1'}^{I_1 I_5} C_{k_2'}^{I_2 I_6} c_{k_1} c_{k_2} \partial_{k_1'}^c \partial_{k_2'}^c \\ &- 3 R_{J I_4}^{k_1} F_{I_3 I_4} C_{k_2}^{I_1 I_2} c_{k_1} \partial_{k_2}^c - 4 (F_{I_3 I_4} F_{I_2 I_5} + \frac{1}{4} A_{I_3 I_4} A_{I_2 I_5}) F_{I_1 I_6} R_{\{J I_4}^{k_1} R_{I_5 I_6\} a s}^{k_2} c_{k_1} c_{k_2} \\ &+ (3 F_{I_3 I_4} F_{I_2 I_5} + \frac{1}{4} A_{I_3 I_4} A_{I_2 I_5}) R_{\{J I_4}^{k_1} R_{I_5 I_6\} a s}^{k_2} C_{k_1'}^{I_1 I_6} c_{k_1} c_{k_2} \partial_{k_1'}^c \left. \right] \\ &+ J_K^{I_1 I_2} b_K [(F_{I_1 I_3} F_{I_2 I_4} + \frac{1}{4} A_{I_1 I_3} A_{I_2 I_4}) R_{I_3 I_4}^k c_k - 2 R_{I_3 I_4}^{k_1} F_{I_1 I_3} C_{k_2}^{I_2 I_4} c_{k_1} \partial_{k_2}^c \\ &- R_{I_3 I_4}^k C_{k_1}^{I_2 I_3} C_{k_2}^{I_1 I_4} c_k \partial_{k_1}^c \partial_{k_2}^c + C_k^{I_2 I_1} \partial_k^c] \left. \right\} |\mathbf{F}(a, c, b) \rangle. \quad (32) \end{aligned}$$

If one likes to look at the lowest order of this infinite hierarchy, one can interpret the functional sources and contractions as destruction and creation operators and thus as ‘images’ of the original field operators. Thus a Feynman like interpretation of the terms is possible. Nevertheless, the correct evaluation has to use projections and thereby the full matrix hierarchy. For a brief account on the physical interpretation of the result we use the former. For example, the $c \partial^c$ terms constitute a positronium current. A decay process has $c \partial^b$ and a creation process $b \partial^c$ as functional sources. The coupling of a positronium current to the boson field is given by $c \partial^c \partial^b$ or $b c \partial^c$. In spite of the stable bound states, which were derived from the self-interacting fermion fields, the mapping reproduces all the necessary effects as decay and creation. As was mentioned above, the solutions $C_k^{I_1 I_2}$ and their duals $R_{I_1 I_2}^k$ contain not only bound states, but also scattering states. Hence there is the possibility of dissociation into free electrons and positrons. Because of the restriction to the even part of the hierarchy, we have no dynamical theory for the free electrons and positrons, which nevertheless could be obtained by the complete mapping. Higher order terms correspond to quantum effects. The concrete elaboration of the formal result is currently under investigation [17]. One should be able to calculate numerical data, as the decay constant etc.

2 Generating Functionals and Clifford Algebras

Because of the very compact and easy to handle notation of the functional space, one would like to better understand this construction. In the above method, only formal aspects of the functional states were used. The manipulations as normal ordering and so on were motivated by QFT arguments. This section is devoted to give a first account on a pure geometric view of field quantization involved in such states. It will be shown, that it is not very difficult to give the functional states a geometric meaning, but the heuristic definition of composites seems to have no such foundation.

2.1 New Transition into the Functional Space

If we want to establish a connection between Clifford algebras and conventional QFT's, we have to search for a common structure. This structure is provided by the generating functionals. Despite their introduction as an auxiliary or mere book-keeping entity, we want to give them an algebraic and geometric meaning by the identification with Clifford algebras. Therefore we look at the generating functionals in a new way. Only if the geometric interpretation fits somehow into the QFT framework, this sort of consideration is a reasonable one. For brevity we restrict ourselves to fermions, but bosons can be treated along the same lines too. We define the fermionic functional as

$$\begin{aligned} |\mathbf{A}(a, j) > &:= \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_n(a) j_{I_1} \dots j_{I_n} |0 >_f \\ \tau_n^t(a) = \tau_n(a) &= i^{-n} {}_f \langle 0 | \partial_{I_n} \dots \partial_{I_1} | \mathbf{A}(a, j) > \\ &:= \langle 0 | \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) | a > . \end{aligned} \quad (33)$$

The anticommuting sources j_I build up an infinite dimensional Grassmann algebra. The commutation relations are

$$\{j_{I_1}, j_{I_2}\}_+ = 0. \quad (34)$$

Thus we may define in a formal manner the linear space spanned by this sources

$$\mathcal{V} := \langle j_{I_1}, \dots, j_{I_n}, \dots \rangle. \quad (35)$$

This set is not countable because of the occurrence of the space time coordinates in the index I . Nevertheless, we proceed with the formal construction. We expect this sort of algebra not to exist, because of the renormalization problem of QFT's as for example mass and charge renormalization in QED. However by this analogy we want to establish Clifford algebraic methods in the field theoretic context. This methods should then give hints how to construct finite field theories, which would then also exist as a Clifford algebra.

The coefficients of the generating elements j_I of the Grassmann algebra are complex valued generalized functions, eg. distributions (τ -functions (33)). Thus we have to build an infinite dimensional Grassmann algebra over the complex valued generalized functions as

$$\Lambda \mathcal{V} := \oplus_n \Lambda^n \mathcal{V} = D \oplus \mathcal{V} \oplus \Lambda^2 \mathcal{V} \oplus \dots \quad (36)$$

with D the space of \mathbb{C} -valued generalized functions. Because even the product of such functions is in general not defined, one should restrict the domain or has to smear out the fields. Because of our formal treatment, we do not discuss this topics. Obviously we then have

$$|\mathbf{A}(a, j) \rangle \in \Lambda\mathcal{V} \quad (37)$$

and the functional states are formal elements of the above Grassmann algebra.

If we want to translate the ‘projection’ of components out of the functional states, we have to define contractions. They are nothing but covectors to the generating elements j_I , which may be called vectors. A vector is thus an element of Grassmann grade one and a covector is a one form. It is natural to choose

$$(\partial_{I_1} j_{I_2}) := \delta_{I_1 I_2}. \quad (38)$$

With this definition, we are immediately able to follow Chevalley [18] and build the Clifford algebra as a deformation of the Grassmann algebra. Thus we define

$$e_I := \partial_I + j_I \quad (39)$$

as Chevalley did for vectors over \mathbf{R} or \mathbf{C} by $e_x := \partial_x + L_x$. Here ∂_x is the usual derivation and L_x is the exterior (Grassmann) product. In our case we get

$$\begin{aligned} \{e_{I_1}, e_{I_2}\}_+ &= 2\delta_{I_1 I_2} \\ e_I^2 &= 1. \end{aligned} \quad (40)$$

Of course this equation is singular, and has to be interpreted as an integral kernel. We have thus constructed the Clifford algebra

$$CL_D(\mathcal{V}, \delta) = CL_D(\langle j_{I_1}, \dots, j_{I_n}, \dots \rangle, \delta) \quad (41)$$

as algebra over the \mathbb{C} -valued generalized functions, see for an example [19]. As a consequence, the functional states are also elements of this Clifford algebra

$$|\mathbf{A}(a, j) \rangle \in CL_D(\mathcal{V}, \delta). \quad (42)$$

The next task is to find a representation of the field operators. Because of the operator character one has to expect them in the space of endomorphisms of the Grassmann algebra. Thus we have

$$\Psi_I \in \text{End}(\Lambda\mathcal{V}). \quad (43)$$

It is by no means clear, that Ψ_I should be contained in the Clifford algebra, which is only a subalgebra of $\text{End}(\Lambda\mathcal{V})$. But this can be concluded from the canonical commutation relations. We have to notice, that the commutator is (Grassmann) grade lowering by two. Thus from the canonical commutation relations

$$\{\Psi_{I_1}, \Psi_{I_2}\}_+ = A_{I_1 I_2} \quad (44)$$

we see that Ψ_I is in the Clifford subalgebra of $End(\Lambda\mathcal{V})$. With the Ansatz

$$\Psi_{I_1} \approx \frac{1}{i}\partial_{I_1} + \frac{i}{2}A_{I_1 I_2}j_{I_2} \quad (45)$$

one can derive the commutator quite easily. In fact this is the formula one got with the Baker–Campbell–Hausdorff formula of (left) derivation of the functional state with respect to j_I (Ψ_{I_1} is understood here as multiplication operator inside the τ -functions, from the left)

$$\partial_{I_1}|\mathbf{A}(a, j) > = \frac{\partial}{\partial j_{I_1}}e^{i\tau j}|0 >_f = (i\Psi_{I_1} + \frac{1}{2}A_{I_1 I_2}j_{I_2})|\mathbf{A}(a, j) > . \quad (46)$$

The same consideration is possible for boson fields also. In the same manner we may define the right multiplication of Ψ_I to the $\tau_n(a)$ functions, which is given by

$$< 0|\mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n})\Psi_I|a > = < 0|\frac{1}{i^n}\partial_{I_n} \dots \partial_{I_1}(\frac{1}{i}\partial_I - \frac{i}{2}A_{II'}j_{I'})|\mathbf{A}(a, j) > . \quad (47)$$

Therewith we can formulate the identities

$$\begin{aligned} < 0|\Psi_{J_1} \dots \Psi_{J_s} B_{L_1} \dots B_{L_r} \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m})|a > = \\ & \frac{1}{i^n} f < 0|\partial_{K_1}^b \dots \partial_{K_m}^b \partial_{I_n} \dots \partial_{I_1} \times \\ & \times (\frac{1}{i}\partial_{L_1}^b + \frac{1}{2}C_{L_1 L'_1} b_{L'_1}) \dots (\frac{1}{i}\partial_{L_r}^b + \frac{1}{2}C_{L_r L'_r} b_{L'_r}) \times \\ & \times (\frac{1}{i}\partial_{J_s} + \frac{i}{2}A_{J_s J'_s} j_{J'_s}) \dots (\frac{1}{i}\partial_{J_1} + \frac{i}{2}A_{J_1 J'_1} j_{J'_1})|\mathbf{A}(a, j, b) > \end{aligned} \quad (48)$$

and

$$\begin{aligned} < 0|\mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m})|a > \Psi_{J_1} \dots \Psi_{J_s} B_{L_1} \dots B_{L_r} = \\ & \frac{1}{i^n} f < 0|\partial_{K_1}^b \dots \partial_{K_m}^b \partial_{I_n} \dots \partial_{I_1} \times \\ & \times (\frac{1}{i}\partial_{L_1}^b - \frac{1}{2}C_{L_1 L'_1} b_{L'_1}) \dots (\frac{1}{i}\partial_{L_r}^b - \frac{1}{2}C_{L_r L'_r} b_{L'_r}) \times \\ & \times (\frac{1}{i}\partial_{J_s} - \frac{i}{2}A_{J_s J'_s} j_{J'_s}) \dots (\frac{1}{i}\partial_{J_1} - \frac{i}{2}A_{J_1 J'_1} j_{J'_1})|\mathbf{A}(a, j, b) > \end{aligned} \quad (49)$$

with

$$\begin{aligned} A_{I_1 I_2} &:= \{\Psi_{I_1}, \Psi_{I_2}\}_+ \\ C_{K_1 K_2} &:= [B_{K_1}, B_{K_2}]_- . \end{aligned} \quad (50)$$

The proof can be done by iteration of single applications of Ψ_I and B_K to the left and right. Such formulas were derived in [14, 8, 4] ■.

By means of these identities one can get in a very easy way the functional Schrödinger equation. Because of

$$\begin{aligned}
i\partial_0\tau_{nm}(a) &= i\partial_0 < 0 | \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m}) | a > \\
&= < 0 | \left[H[\Psi, B], \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m}) \right] | a > \\
&= < 0 | H[\Psi, B] \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m}) | a > \\
&\quad - < 0 | \mathcal{A}(\Psi_{I_1} \dots \Psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m}) H[\Psi, B] | a >
\end{aligned} \tag{51}$$

we have the fields inside the Hamiltonian one time left and one time right of the (anti) symmetrized blocks. Thus we get the transition

$$H[\Psi, B] \implies H[j, b, \partial, \partial^b] \tag{52}$$

by subtracting the two different substitutions of $H[\Psi, B]$ with $\Psi \rightarrow \partial \pm A_j$, $B \rightarrow \partial^b \pm Cb$ as

$$H[j, b, \partial, \partial^b] = H[\Psi, B] \Big|_{\substack{\Psi \approx \partial + A_j \\ B \approx \partial + Cb}} - H[\Psi, B] \Big|_{\substack{\Psi \approx \partial - A_j \\ B \approx \partial - Cb}}. \tag{53}$$

The resulting functional equation is equivalent to (22). Besides the enormous shortened calculation, we have established that the commutators of field quantized theories are nothing but the metric of the desired Clifford algebra in the functional space. One yields therefore

$$|\mathbf{A}(a, j, b) > \in CL_D(\mathcal{V} \oplus \mathcal{W}, A \oplus C) \tag{54}$$

where \mathcal{V} is the fermion (vector) space and \mathcal{W} is the boson (vector) space spanned by j and b . A and C are the corresponding metrics.

In a previous paper we investigated the vertex normal ordering, which has to be performed in a QFT with nonlinear self interaction [20]. There we found the connection between this reordering and the choice of a specific Clifford algebra out of a class of isomorphic ones. Even our composite calculation forced us to introduce a normal ordering. Thus we have to look for a generalization of the above transition. This is necessary, because if we choose the normal ordered QFT to be finite, then the not normal ordered one has singular interaction terms and vice versa. To avoid such ill defined steps in the calculation, we have to find a direct transition to the normal ordered equation.

So far, we had only symmetric $A_{I_1 I_2}$ in the Ansatz (45) of the fermion operators. We allow now arbitrary bilinearforms B , which we can split B into a symmetric part A and an antisymmetric part F . Since only A and F occur in functional formulas a confusion with the bosonfields also denoted B is impossible. We can then introduce left and right multiplications as

$$\hat{\Psi}_{I_1}^{L/R} \approx \frac{1}{i} \partial_{I_1} \pm \frac{i}{2} A_{I_1 I_2} j_{I_2} + i F_{I_1 I_2} j_{I_2}. \tag{55}$$

The asymmetric choice of factors $1/2$ is due to physical considerations about the normal ordering, which was introduced above by $d_I := \partial_I - F_{II'} j_{I'}$. Of course this choice also leads to the correct prefactors in the normal ordered functional equation (67). An even better choice would have changed

the commutator to $2A$. The i factors are relics of the definition of the fermionic functionals as $\exp(i\tau j)$ and could be doped. Of course we choose (55) for the reason of comparison with previously obtained results.

Much more important is the occurrence of the *same* sign (+) in front of the $F_{I_1 I_2}$. That is, we have in the antisymmetric part a strong asymmetry in the conjugation structure. This sort of asymmetry was studied to a considerable length in [21]. Hence we remark, that the right and left action is distinguished by the *commutator*, while the propagator acts both times in the same way. This corresponds to a non grade invariant involution, which has the propagator as scalar part (in the sense of Grassmann grade). Of course this stems from the use of the naive Grassmann wedge built up from antisymmetric Grassmann sources of definite Grassmann grade, i.e. the j_I . Because the injection $\Lambda\mathcal{V} \rightarrow CL(\mathcal{V}, A)$ is by no means unique, the normal ordering is nothing but the transformation to the correct, that is the physical reasonable, wedge. For a detailed account on that see [20]. Thus the *propagator* fixes the involution and therefore the grading structure. This choice is thus nothing but the choice of the appropriate physical vacuum, which is here represented as the part with (Grassmann) grade zero, belonging to the ‘physical’ choice from above.

We have the commutators for the bosons

$$\begin{aligned} [\partial_{K_1}^b - \frac{1}{2}C_{K_1 K_1'} b_{K_1'}, \partial_{K_2}^b - \frac{1}{2}C_{K_2 K_2'} b_{K_2'}]_- &= C_{K_1 K_2} \\ [\partial_{K_1}^b + \frac{1}{2}C_{K_1 K_1'} b_{K_1'}, \partial_{K_2}^b + \frac{1}{2}C_{K_2 K_2'} b_{K_2'}]_- &= -C_{K_1 K_2} \\ [\partial_{K_1}^b \pm \frac{1}{2}C_{K_1 K_1'} b_{K_1'}, \partial_{K_2}^b \mp \frac{1}{2}C_{K_2 K_2'} b_{K_2'}]_- &= 0, \end{aligned} \quad (56)$$

while we have for the normal ordered fermions

$$\begin{aligned} \{\frac{1}{i}\partial_{I_1} \pm \frac{i}{2}A_{I_1 I_1'} j_{I_1'} + iF_{I_1 I_1'} j_{I_1'}, \frac{1}{i}\partial_{I_2} \pm \frac{i}{2}A_{I_2 I_2'} j_{I_2'} + iF_{I_2 I_2'} j_{I_2'}\}_+ &= \\ \pm \frac{1}{2}A_{I_2 I_2'} \{\partial_{I_1}, j_{I_2'}\}_+ + F_{I_2 I_2'} \{\partial_{I_1}, j_{I_2'}\}_+ \pm \frac{1}{2}A_{I_1 I_1'} \{\partial_{I_2}, j_{I_1'}\}_+ + F_{I_1 I_1'} \{\partial_{I_2}, j_{I_1'}\}_+ &= \pm A_{I_1 I_2} \\ \{\frac{1}{i}\partial_{I_1} \pm \frac{i}{2}A_{I_1 I_1'} j_{I_1'} + iF_{I_1 I_1'} j_{I_1'}, \frac{1}{i}\partial_{I_2} \mp \frac{i}{2}A_{I_2 I_2'} j_{I_2'} + iF_{I_2 I_2'} j_{I_2'}\}_+ &= 0. \end{aligned} \quad (57)$$

If one had chosen $\pm F$ as for the commutator, one would not have reached the goal of left and right multiplication. This is, because of the mixed commutator, which would have not been zero. But left and right multiplications have to be commutative operations in any associative algebra.

We are now able to calculate in a direct manner the (fermion) normal ordered functional Hamiltonian from $H[\Psi, B]$. Therefore we have to build

$$\hat{H}[j, b, \partial, \partial^b] = H[\Psi, B] \Big|_{\substack{\Psi_I \approx \frac{1}{i}\partial_I + \frac{i}{2}A_{II'} j_{I'} + iF_{II'} j_{I'} \\ B_K \approx \partial_K^b + \frac{i}{2}C_{KK'} b_{K'}}} - H[\Psi, B] \Big|_{\substack{\Psi_I \approx \frac{1}{i}\partial_I - \frac{i}{2}A_{II'} j_{I'} + iF_{II'} j_{I'} \\ B_K \approx \partial_K^b - \frac{i}{2}C_{KK'} b_{K'}}} \quad (58)$$

We have according to (16) four terms T_i . They will be now separately considered

$$T_1 = \frac{1}{2}A_{I_1 I_3} D_{I_3 I_2} \Psi_{I_1} \Psi_{I_2}$$

$$\begin{aligned}
&\approx \frac{1}{2}A_{I_1 I_3}D_{I_3 I_2}\left[\left(\frac{1}{i}\partial_{I_1} + \frac{i}{2}A_{I_1 I'_1}j_{I'_1} + iF_{I_1 I'_1}j_{I'_1}\right)\left(\frac{1}{i}\partial_{I_2} + \frac{i}{2}A_{I_2 I'_2}j_{I'_2} + iF_{I_2 I'_2}j_{I'_2}\right)\right. \\
&\quad \left.-\left(\frac{1}{i}\partial_{I_1} - \frac{i}{2}A_{I_1 I'_1}j_{I'_1} + iF_{I_1 I'_1}j_{I'_1}\right)\left(\frac{1}{i}\partial_{I_2} - \frac{i}{2}A_{I_2 I'_2}j_{I'_2} + iF_{I_2 I'_2}j_{I'_2}\right)\right] \\
&= \frac{1}{2}A_{I_1 I_3}D_{I_3 I_2}\left[\left\{\left(\frac{1}{i}\partial_{I_1} + \frac{i}{2}A_{I_1 I'_1}j_{I'_1}\right)\left(\frac{1}{i}\partial_{I_2} + \frac{i}{2}A_{I_2 I'_2}j_{I'_2}\right)\right.\right. \\
&\quad \left.-\left(\frac{1}{i}\partial_{I_1} - \frac{i}{2}A_{I_1 I'_1}j_{I'_1}\right)\left(\frac{1}{i}\partial_{I_2} - \frac{i}{2}A_{I_2 I'_2}j_{I'_2}\right)\right\} \\
&\quad + \left\{\left(\frac{1}{i}\partial_{I_1} + \frac{i}{2}A_{I_1 I'_1}j_{I'_1}\right)iF_{I_2 I'_2}j_{I'_2} + iF_{I_1 I'_1}j_{I'_1}\left(\frac{1}{i}\partial_{I_2} + \frac{i}{2}A_{I_2 I'_2}j_{I'_2}\right)\right. \\
&\quad \left.-\left(\frac{1}{i}\partial_{I_1} - \frac{i}{2}A_{I_1 I'_1}j_{I'_1}\right)iF_{I_2 I'_2}j_{I'_2} - iF_{I_1 I'_1}j_{I'_1}\left(\frac{1}{i}\partial_{I_2} - \frac{i}{2}A_{I_2 I'_2}j_{I'_2}\right)\right\} \\
&\quad \left.+ \left\{-F_{I_1 I'_1}F_{I_2 I'_2}j_{I'_1}j_{I'_2} + F_{I_1 I'_1}F_{I_2 I'_2}j_{I'_1}j_{I'_2}\right\}\right]. \tag{59}
\end{aligned}$$

With help of symmetry arguments on ∂ and j as the antisymmetry of the AD factor in $I_1 I_2$ we have

$$\begin{aligned}
T_1 &= \frac{1}{2}A_{I_1 I_3}D_{I_3 I_2}\left\{4\frac{i}{2}\frac{1}{i}A_{I_1 I'_1}j_{I'_1}\partial_{I_2} + 4i\frac{i}{2}A_{I_1 I'_1}F_{I_2 I'_2}j_{I'_1}j_{I'_2} + 0\right\} \\
&= D_{I_1 I_2}j_{I_1}\partial_{I_2} - D_{I_1 I_3}F_{I_3 I_2}j_{I_1}j_{I_2}, \tag{60}
\end{aligned}$$

which yields because of the nature of the propagator and the renormalization discussed in [8]

$$\begin{aligned}
T_1 &= D_{I_1 I_2}j_{I_1}\partial_{I_2} - Z_{I_1 I_3}F_{I_3 I_2}j_{I_1}j_{I_2} \\
Z_{I_1 I_3} &:= -\delta m\gamma_{\alpha_1 \alpha_3}^0 \delta_{\Lambda_1 \Lambda_3} \delta(\mathbf{r}_1 - \mathbf{r}_3). \tag{61}
\end{aligned}$$

The term T_2 is given as

$$\begin{aligned}
T_2 &= \frac{1}{2}A_{I_1 I_3}W_{I_3 I_2}^K B_K \Psi_{I_1} \Psi_{I_2} \\
&\approx \frac{1}{2}A_{I_1 I_3}W_{I_3 I_2}^K \left[(\partial_K^b + \frac{1}{2}C_{KK'}b_{K'}) (\dots)(\dots)\right. \\
&\quad \left.-(\partial_K^b - \frac{1}{2}C_{KK'}b_{K'}) (\dots)(\dots)\right] \\
&= \frac{1}{2}A_{I_1 I_3}W_{I_3 I_2}^K \left[\partial_K^b \{(\dots)(\dots) - (\dots)(\dots)\}\right. \\
&\quad \left.+ \frac{1}{2}C_{KK'}b_{K'} \{(\dots)(\dots) + (\dots)(\dots)\}\right]. \tag{62}
\end{aligned}$$

The first term is just the same as before, while the second yields the terms

$$\begin{aligned}
2. \text{ brace } \{ \} &= -2\partial_{I_1}\partial_{I_2} - 2\frac{1}{4}A_{I_1 I'_1}A_{I_2 I'_2}j_{I'_1}j_{I'_2} \\
&\quad + 4F_{I_1 I'_1}j_{I'_1}\partial_{I_2} - 2F_{I_1 I'_1}F_{I_2 I'_2}j_{I'_1}j_{I'_2}. \tag{63}
\end{aligned}$$

Further more we may use (17) to obtain

$$\frac{1}{4}A_{I_1 I_3}W_{I_3 I_2}^K C_{KK'}b_{K'} = -\frac{1}{4}A_{I_1 I_3}C_{K'K}b_{K'}W_{I_3 I_2}^K \quad (64)$$

$$\begin{aligned} &= -\frac{1}{2}A_{I_1 I_3}A_{I_3 I_4}J_{K'}^{I_4 I_2}b_{K'} \\ &= -\frac{1}{2}J_K^{I_1 I_2}b_K \end{aligned} \quad (65)$$

and obtain as result

$$\begin{aligned} T_2 = & W_{I_1 I_2}^K [j_{I_1} \partial_{I_2} - F_{I_2 I_2'} j_{I_1} j_{I_2'}] \partial_K^b \\ & + J_k^{I_1 I_2} b_K [\partial_{I_1} \partial_{I_2} - 2F_{I_1 I_1'} j_{I_1'} \partial_{I_2} \\ & + (F_{I_1 I_1'} F_{I_2 I_2'} + \frac{1}{4}A_{I_1 I_1'} A_{I_2 I_2'}) j_{I_1'} j_{I_2'}]. \end{aligned} \quad (66)$$

The third term is lengthy and will be omitted, as the scheme is clear now. The last term runs along the lines of T_1 and has not changed at all from (22). Our result is then

$$\begin{aligned} E_{0a}|\mathbf{F}(a, j, b) > = & \left\{ D_{I_1 I_2} j_{I_1} \partial_{I_2} - Z_{I_1 I_3} F_{I_3 I_2} j_{I_1} j_{I_2} \right. \\ & + W_{I_1 I_2}^K [j_{I_1} \partial_{I_2} - F_{I_2 I_2'} j_{I_1} j_{I_2'}] \partial_K^b \\ & + J_k^{I_1 I_2} b_K [\partial_{I_1} \partial_{I_2} - 2F_{I_1 I_1'} j_{I_1'} \partial_{I_2} \\ & + (F_{I_1 I_1'} F_{I_2 I_2'} + \frac{1}{4}A_{I_1 I_1'} A_{I_2 I_2'}) j_{I_1'} j_{I_2'}] \\ & + U_J^{I_1 I_2 I_3} j_J [\partial_{I_1} \partial_{I_2} \partial_{I_3} - 3F_{I_3 I_4} j_{I_4} \partial_{I_2} \partial_{I_1} \\ & + (3F_{I_3 I_4} F_{I_2 I_5} + \frac{1}{4}A_{I_3 I_4} A_{I_2 I_5}) j_{I_4} j_{I_5} \partial_{I_1} \\ & - (F_{I_3 I_4} F_{I_2 I_5} F_{I_1 I_6} + \frac{1}{4}A_{I_3 I_4} A_{I_2 I_5} A_{I_1 I_6}) j_{I_4} j_{I_5} j_{I_6}] \\ & \left. + L_{K_1 K_2} b_{K_1} \partial_{K_2}^b \right\} |\mathbf{F}(a, j, b) > . \end{aligned} \quad (67)$$

which coincides with (8.59) in [4] and (63) in [8]. So we have derived the normal ordered functional equation in one single step by Clifford algebraic considerations. No normal ordering problems, as a vertex normal ordering, was necessary. No singular intermediate equation has occurred.

2.2 Criticism of the Heuristic Composite Definition

Our next step should be to define the composite. But if we want to translate simply the concept of taking the diagonal part, we do not succeed. Of course it is possible to extract the diagonal equations quite easily in the Clifford algebraic picture too, but there are no geometric arguments to do so. Thus we have to reject this sort of considerations in the algebraic framework. This is a new argument against Bethe–Salpeter like equations.

The next problem is, that even in the QFT formalism one can not be satisfied with such “hard core” solutions. There should be a polarization cloud consisting of infinite many particle–anti-particle pairs. A translation of such a polarization cloud into the geometric picture would result

in a highly nonlinear equation. The same problem occurs if one likes to use selfconsistency arguments. This stems from the fact that a part of the renormalized particle is built up from “vacuum” polarization. The Clifford algebraic picture unveils thus this wrong point of view. The physical “vacuum” was simply wrong chosen by the use of the naive (Grassmann) grading. The injection of the Grassmann grading into the Clifford algebra has carefully to be accounted for and the “vacuum” is the scalar part of the nonsymmetric involution [20, 21].

New C^* -algebraic investigations [22] give a hint, that a q -deformed theory *for the composites* may solve this problem. In such cases one has to choose q as a root of unity. The representations of this algebras are reducible and offer some freedom to build up new states. A Clifford algebraic formulation of this will be given elsewhere. One can also remark that in general composite particle dynamics can not be described by a (usual, i.e. canonically quantized) quantum field theory at all. The use of q -deformations restricts the situation too, which may be useful in some situations but which does not cover the whole range of composite particle physics as will be shown elsewhere. So, as a result there are *no* current methods to define QFT bound states in an algebraic framework, but the clear Clifford algebraic formulation gives some hints and provides the tools how to build them.

3 Summary

In the first section we gave an outline of the well developed method of weak mapping. This method provides a most useful and consistent approach to QFT via generating functionals. It also offers the best available mechanism to deal with composites including their quantum properties and dynamics. As an example we treated spinor QED in Coulomb gauge to demonstrate the method. The considerations are fully contained in the QFT picture of matrix elements, operators, field quantization, states and so on. The occurring problems might be handled by heuristic methods as for example the renormalization procedure.

In the second section we tried to give a new interpretation of field quantization. This was motivated by Clifford algebraic methods. As a surprising fact we had to distinguish between several isomorphic Clifford algebras parameterized by the propagator of the theory. We can summarize this in the following conjecture:

*Field quantization is nothing but the introduction of geometry into the functional space. The **commutator** chooses the desired Clifford algebra, the **propagator** fixes the specific representant of the isomorphic algebras.*

We have thus learned to use geometric arguments instead of a heuristic ‘quantization’ of classical theories. The geometric reasoning did provide us a direct way to the normal ordered functional equation without passing through ill defined intermediate steps.

The heuristic composite definition was rejected by this sort of geometric arguments. But the method itself gives a hint how to proceed with composite definitions. This will be done in detail in another work.

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